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LETTER TO THE EDITOR

General Poissonian model of diffusion in chaotic components

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Abstract. We extend our recent study of diffusion in strongly chaotic systems ('the random model') to the general systems of mixed-type dynamics, including especially KAM systems, regarding the diffusion in chaotic components. We do this by introducing a Poissonian model as in our previous random model describing the strongly chaotic systems, except that now we allow for different *a priori* probabilities in different cells of the discretized phase space (surface of section). Thus the concept of greyness (of cells), denoted by g, such that $0 \leqslant g \leqslant 1$, is introduced, as is its distribution $w(g)$. We derive the relationship between the dynamical property, namely the (normalized) fraction of chaotic component $\rho(j)$ as a function of discrete time j, and $w(g)$. We predict again the universal scaling law, namely that for any $w(g)$, the chaotic fraction $\rho(j)$ is a function of j/N only, and not separately of j and N, where N is the number of cells of equal size $1/N$. The random model of exponential $1 - \rho(j) = \exp(-j/N)$ is reproduced if all cells have $g = 1$, i.e. $w(g) = \delta(1 - g)$. We argue that in two-dimensional systems, at any finite N, $w(g)$ is non-trivial due to the fractal dimension of the boundary of the chaotic component, but is such that it goes to $\delta(1-g)$ as $N \to \infty$, whilst in systems with three or more degrees of freedom $w(g)$ has a well defined limit with non-zero values also at $g < 1$. This is due to the existence of the Arnold web. We suggest how—through our formalism—one can calculate the Lebesgue measure of the chaotic component at each finite discretization, whose limit exists for $N \to \infty$. Our findings are verified and illustrated for two- and three-dimensional billiards.

In this work we put forward a method of calculating the Lebesgue measure of chaotic components in Hamiltonian systems of mixed-type dynamics. We divide the phase space (actually the surface of section (SOS)) into a sufficiently large number of cells of equal relative measure, such that the number of cells containing points of the chaotic component is equal to N, and each cell has the relative Lebesgue measure equal to $a = 1/N$. We assume that N is sufficiently large, although this is not a very essential assumption. We start an orbit in one of the cells, follow the discrete orbit as it evolves with the discrete time j (= number of iterations of the Poincaré mapping on the SOS), and calculate the total relative Lebesgue measure of the cells visited up to the time j, denoted by $\rho_2(j)$. We refer to the limiting value (as $j \to \infty$) of this quantity as *the discrete measure* of the chaotic component. By construction it is based on box counting. We ask the question: Under

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what conditions is the limiting value $\rho_2(\infty)$ equal or close to the relative Lebesgue measure of the chaotic component? Clearly, in two degrees of freedom we expect convergence of $\rho_2(\infty)$ to the Lebesgue measure as $N \to \infty$. However, the convergence can be very slow when the fractal dimension of the boundary of the chaotic region is large, namely close to two. In fact, this very circumstance is quite typical. In three or more degrees of freedom there exists the Arnold web (Chirikov 1979), which is dense in the phase space and on the SOS, and therefore at any N each cell contains some points of the Arnold web (= chaotic component), so that in three or more degrees of freedom we *always* have $\rho_2(\infty) = 1$ for any N, implying that the discrete measure of the chaotic component is one for any N. So, in this case the difficult question arises as to how to measure/determine the Lebesgue measure of the (connected) chaotic region (Arnold web). In KAM-type systems we know that the relative Lebesgue measure of the chaotic component must be less than one, because the KAM theorem guarantees that the measure of invariant tori is not only larger than zero, but is even close to one, such that the measure of the complement (excluded by the KAM inequality) goes to zero with the perturbation parameter. We offer an answer to this question and develop the method of how to achieve this. In the following we shall denote the normalized discrete measure as a function of discrete time j by $\rho(j) = \rho_2(j)/\rho_2(\infty)$. Thus, by construction $\rho(j) \to 1$ as $j \to \infty$.

First we make some of the above statements more precise. Hamiltonian systems with mixed-type dynamics are generic, the most typical, and therefore the most important ones. They are well described by the KAM scenario. It is the KAM theorem mentioned above that proves that the set of invariant tori has positive measure, whose complement is small with the perturbation parameter (Kolmogoroff 1954, Arnold 1963, Moser 1962, Benettin *et al* 1984, Gutzwiller 1990). However, it (the KAM theorem) does not make any statement about the initial conditions that belong to the complementary set which is born out of not sufficiently irrational tori of the integrable part. Typically, the orbits there are chaotic and their chaoticity could be defined and identified by the positivity of the largest Lyapunov exponent, which is certainly sufficient but not a necessary criterion†. The fundamental open problem in the mathematics of nonlinear dynamical systems with mixed dynamics is to prove that the chaotic component has in fact positive measure. This is the so-called coexistence problem. For a nice exposition see the review paper by Strelcyn (1991).

We can define a chaotic component as the set containing a dense chaotic orbit, which is thus assumed to be an indecomposable invariant component (topologically transitive). The quantity we seek is the (relative) Lebesgue measure of the chaotic component. Clearly, the closure of a chaotic component has the relative Lebesgue measure equal to unity in three or more degrees of freedom, because the Arnold web is dense in phase space, whilst in two degrees it can be smaller than one.

In physics we have no serious doubts about the positivity of the measure of the chaotic component, relying on heuristic arguments, suggesting that we actually assume positivity. The highly non-trivial question is then how to (numerically) calculate the symplectic (invariant and ergodic) Lebesgue measure of the chaotic component.

We have approached this problem in a recent extensive work (Robnik *et al* 1997, henceforth referred to as (I)), where we have developed *the random model*: each cell (of relative Lebesgue measure $a = 1/N$, where N is the number of cells containing the points of a dense chaotic orbit) can be visited randomly, without any correlations (with the previous visits) whatsoever, with the same *a priori* probability a. The result is, for sufficiently large

† For example, in non-rational plane polygonal billiards all Lyapunov exponents are strictly zero (Sinai 1976), and yet they can be ergodic. Strong evidence for this has recently been published by Artuso *et al* (1997).

N, but at fixed and finite j/N, that the relative discrete measure of the visited cells $\rho(j)$ is exponentially approaching its limiting value unity,

$$\rho(j) = 1 - \exp(-j/N). \tag{1}$$

In (I) we have given the exact solutions of this random model for any j and N; however, the approach to the exponential law given above with increasing N is very fast. In such a case we observe the scaling law, namely that $\rho(j)$ is a function of j/N only, and thus does not depend on j or N separately. In (I) we have also calculated the second moment $S(j)$ and the dispersion $\sigma^2(j)$. The generalized results will be given in the following.

In this work we generalize the ideas and the model by introducing the concept of greyness of the cells: each cell has a certain greyness g, by definition $0 \leqslant g \leqslant 1$, which is proportional to the *a priori* probability of visiting the cell, and thus it is proportional to the relative occupancy number for the cell in the limit $j \to \infty$. This is the only modification of the random model, because we keep on assuming the complete lack of any correlations between the visits of cells, thus assuming a Poisson model with different *a priori* probabilities of visiting different cells. We shall refer to this model as *the generalized random model*, or *the general Poissonian model*.

By $w(g)$ we denote the greyness distribution of cells, assuming that it is defined for any N. Thus $w(g)\,\mathrm{d}g$ is the probability that a given cell has greyness in the interval $(g, g + \mathrm{d}g)$. Of course, $w(g)$ is a normalized probability distribution, $\int_0^1 w(g)\,\mathrm{d}g = 1$.

For sufficiently large time j we then have that the average occupancy number $n(g)$ in a cell of greyness g is proportional to g,

$$n(g) = \alpha g \tag{2}$$

where α is the proportionality factor to be determined. Furthermore, the number of visits Δj that fall in the cells of the greyness interval Δg is simply given by

$$\frac{\Delta j}{\Delta g} = j w(g) \tag{3}$$

and thus it is proportional to j, for large enough j. Our first fundamental equation is then

$$j = \sum_{\Delta g} n(g) \frac{\Delta j}{\Delta g} \Delta g \tag{4}$$

where we sum up over all greyness intervals Δg. Using the above equations (which are in fact definitions), and going over to the infinitesimal intervals $\mathrm{d}g$, we obtain the integral

$$\alpha \int_0^1 g w(g)\,\mathrm{d}g = 1 \tag{5}$$

which determines the value of α, and is equivalent to normalizing the average occupancy number $n(g)$. Thus $n(g)/N$ is identified with the *a priori* Poissonian probability for a cell of greyness g.

Now we want to calculate the expected number $\bar{N}_0$ of non-occupied cells. In each greyness interval Δg we have $w(g)N\Delta g$ cells. The mean number $\bar{m}$ of visits in a cell of such a greyness interval is

$$\bar{m} = \frac{j n(g)}{N}. \tag{6}$$

Now the assumption of Poissonian statistics implies immediately that the probability of having m visits in a cell within the given greyness interval with the average number of visits $\bar{m}$ is equal to

$$P(m, \bar{m}) = \mathrm{e}^{-\bar{m}} \frac{\bar{m}^m}{m!} \tag{7}$$

and therefore, in particular,

$$P(0,\bar{m}) = \mathrm{e}^{-\bar{m}} = \exp\left(-\frac{jn(g)}{N}\right). \tag{8}$$

It is then obvious to write our second fundamental equation

$$\bar{N}_0 = \sum_{\Delta g} P(0,\bar{m}) N w(g) \Delta g \tag{9}$$

which in the limit $\Delta g \to 0$ becomes the integral

$$\bar{N}_0 = \int_0^1 \mathrm{d}g N w(g) \exp\left(-\frac{j}{N}\alpha g\right) \tag{10}$$

and therefore our expected measure of occupied cells, equal to $\rho = 1 - \bar{N}_0/N$, as a function of j/N only, becomes

$$\rho(j) = 1 - \int_0^1 \mathrm{d}g w(g) \exp\left(-\frac{\alpha j}{N} g\right) \tag{11}$$

which is the main result of this paper. Clearly, for any $w(g)$ we have the scaling law that $\rho(j)$ depends on j and N only through j/N, but not separately.

As a first task we reproduce our *random model*, for which we have no greyness, but only black cells, i.e. N cells of equal *a priori* probability $a = 1/N$, so that $g = 1$ for all of them and $w(g) = \delta(1-g)$. Integrating (11) yields exactly (1).

Now we consider the general case of non-trivial $w(g)$. From the Laplace-like transform relationship in (11) we see that the asymptotic behaviour of $\rho(j)$ at large j is dictated by those cells whose greyness g is small. These are of course exactly those cells which are seldom visited. In systems with non-trivial $w(g)$ we, therefore, expect some agreement with the random model (1) for not too large j, whilst at very large j we can find a power law, as will be demonstrated below in a two-dimensional (2D) billiard system. Indeed, if $w(g)$ is a power law at small g,

$$w(g) = Cg^{-\beta} \qquad g \to 0 \tag{12}$$

where $\beta < 1$ due to the integrability (normalizability) of $w(g)$, then $1 - \rho(j)$ is also a power law, $1 - \rho(j/N) \propto (j/N)^{-\gamma}$,

$$\rho(j/N \gg 1) \approx 1 - \frac{C\Gamma(1-\beta)}{(\alpha j/N)^{1-\beta}} \tag{13}$$

so that we have the relation between the exponents β and γ,

$$\beta + \gamma = 1 \tag{14}$$

which can be, of course, phenomenologically (numerically) verified in specific systems (see later).

Since our theory is a statistical model we must give an estimate of the expected statistical fluctuations, i.e. of the dispersion. We do this by considering the situation where j is sufficiently large, $j \gg N$, such that the numbers of empty cells N_0 are uncorrelated and therefore Poissonian distributed, with the expected value $\bar{N}_0$, namely according to the distribution $P(N_0, \bar{N}_0)$ of equation (7). Denoting by $\langle \ldots \rangle$ the statistical average using the Poissonian distribution (7), we find at once that

$$\langle N_0^2 \rangle - \bar{N}_0^2 = \bar{N}_0 \tag{15}$$

and then after integrating over all classes of greyness g, weighted by $Nw(g)$, using the notation $N_1 = N - N_0$ for the number of occupied cells, we calculate

$$\sigma^2(j) = \left\langle \left(\frac{N_1}{N}\right)^2 \right\rangle - \left\langle \frac{N_1}{N} \right\rangle^2 \tag{16}$$

and find

$$\sigma^2(j) = \frac{1-\rho(j)}{N}. \tag{17}$$

This result agrees to leading order with that for the random model (see equation (15) in paper (I), for sufficiently large j/N).

Here the greyness distribution $w(g)$ plays a key role in determining the behaviour of $\rho(j)$. It is a signature of a given chaotic component, at given discretization N. In fact, we shall give numerical evidence that in 2D systems $w(g)$ at finite N behaves as in equation (12), but such that as $N \to \infty$ it converges to $w(g) = \delta(1-g)$. In such a case the relative Lebesgue measure and the discrete measure are identical, although the convergence of the discrete measure to the Lebesgue measure with N might be slow, for example if the fractal dimension of the boundary of the chaotic region is large, close to two. However, in three or more degrees of freedom we have some genuine greyness even when $N \to \infty$, due to the Arnold web, and thus not every box at a given discretization N contributes equally to the Lebesgue measure, so that the limiting relative Lebesgue measure is less than one. Indeed, each cell contributes to the relative Lebesgue measure only the fraction equal to its greyness g, so that in general the relative Lebesgue measure μ is given by

$$\mu = \alpha^{-1} = \int_0^1 g w(g)\, \mathrm{d}g \tag{18}$$

and $0 \leqslant \mu \leqslant 1$ is satisfied. We have reached the important conclusion that by measuring $w(g)$ we can calculate the Lebesgue measure of a given chaotic component, at the given discretization N.

Now we turn to the interpretation of our results in billiard systems. In figure 1 we show the results for the 2D billiard (Robnik 1983, Robnik *et al* 1997 and references therein), with the value of $\lambda = 0.15$. We show the curves $\rho(j)$ on a log–log plot for three different values of N, thereby also demonstrating the scaling law. (Here and in the following, also in figures 1–4, N is the number of cells containing the chaotic orbit.) They follow the exponential (random) law for small j/N, but clearly exhibit power-law behaviour at large j/N, with the exponent $\gamma \approx 0.53$. In figure 2 we show the results for the same billiard at the same three different values for the greyness distribution $w(g)$, on a log–linear plot and the inset with a log–log plot at small g. We clearly observe the power-law behaviour (12) with $\beta \approx 0.46$, which is thus consistent with our theoretical relation (14). It is important to realize that here $w(g)$ contracts down to zero at $g < 1$, building up the delta function spike $w(g) = \delta(1-g)$, eventually, in the limit $N \to \infty$. Thus $w_N(g) \propto N^{-f}$, where $f = 2 - d$ and d is the fractal dimension of the boundary of the chaotic region. Namely, grey cells $g < 1$ in 2D systems are only those that lie on the boundary of the chaotic component. The exponent of this contraction is indeed observed to be roughly $f \approx 0.3$, which is consistent with our estimate of the fractal dimension of the boundary of the chaotic region of about $d \approx 1.7$.

In figure 3 we show the analogous results $\rho(j)$ for the three-dimensional billiard (3D) (Prosen 1997a, b, Robnik *et al* 1997), for $a = -0.1$ and $b = 0$, which is a well pronounced KAM regime. Here one should observe that the curves $\rho(j) = \rho(j/N)$ converge to a single curve, just because the greyness distribution $w(g)$ has a limiting non-trivial shape, which

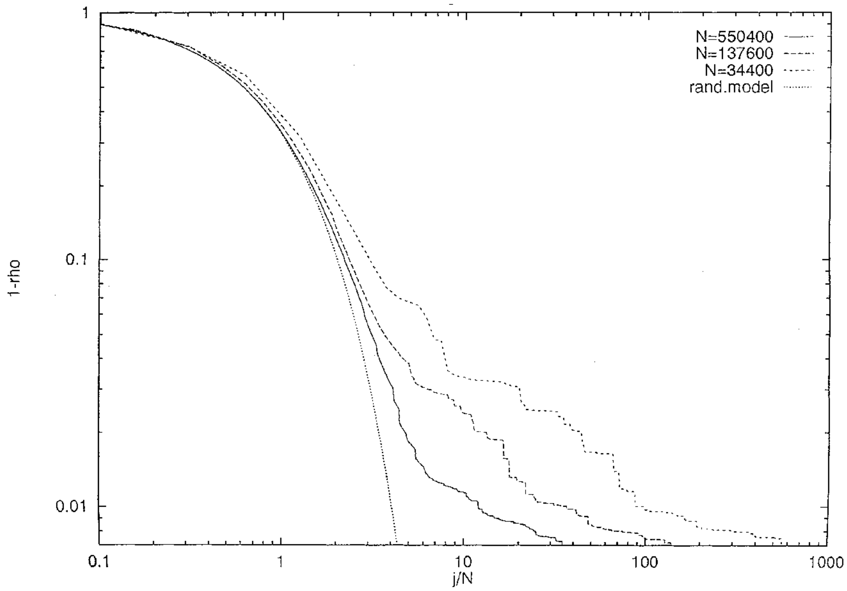

Figure 1. $1 - \rho(j)$ against j/N on a log–log plot, for the 2D billiard (Robnik 1983, $\lambda = 0.15$), for three different values of discretization N, and also for comparison the theoretical curve of the random model, namely $\exp(-j/N)$. It can be seen that $\gamma \approx 0.53$.

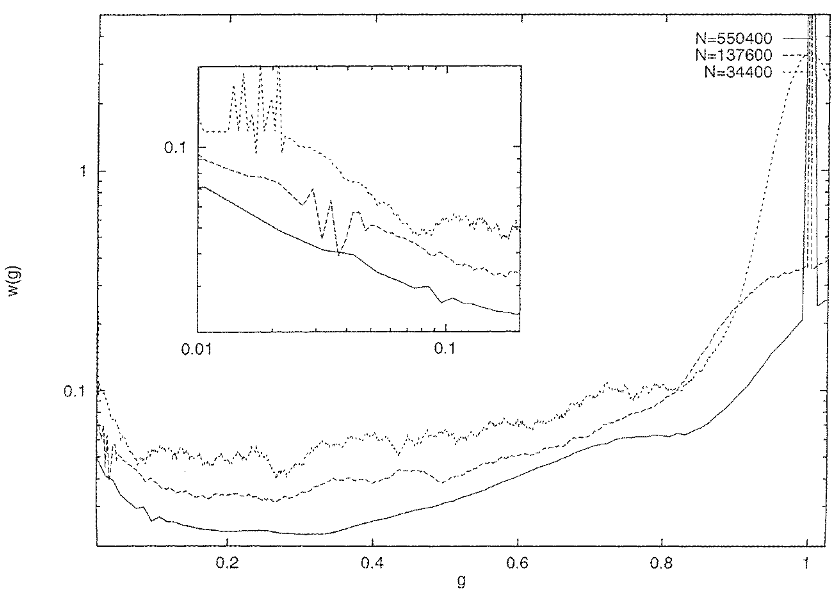

Figure 2. $w(g)$ for the same system as in figure 1, at the same three different values of N, after deconvolution. The smooth background of $w(g)$ clearly decays to zero, with the exponent $f \approx 0.3$. See the text for details. In the inset we show the same curves on a log–log plot. It can be deduced that $\beta \approx 0.46$.

can be seen in figure 4. Thus, in figure 3 we observe the validity of the general scaling law, implied by (11), namely that $\rho(j)$ is only a function of (j/N) but not separately of j and

N, provided $w(g)$ exists in the limit $N \to \infty$. Again, perhaps we have here a power-law behaviour, with $\gamma \approx 0.13$, but this is not so reliably manifested as in the 2D case of figure 1. There are five different values of N. In figure 4 we show the $w(g)$ plots for the same five different values of N. The most important observation here is that now $w(g)$ does not contract to zero at $g < 1$, but has a well defined smooth limit as N increases to infinity. At small g we observe the power law with $\beta \approx 0.48$, which, however, is not in agreement with relation (14). Further numerical work is necessary to determine this behaviour more precisely, but the general relationship between $\rho(j)$ and $w(g)$ described in equation (11) is certainly confirmed.

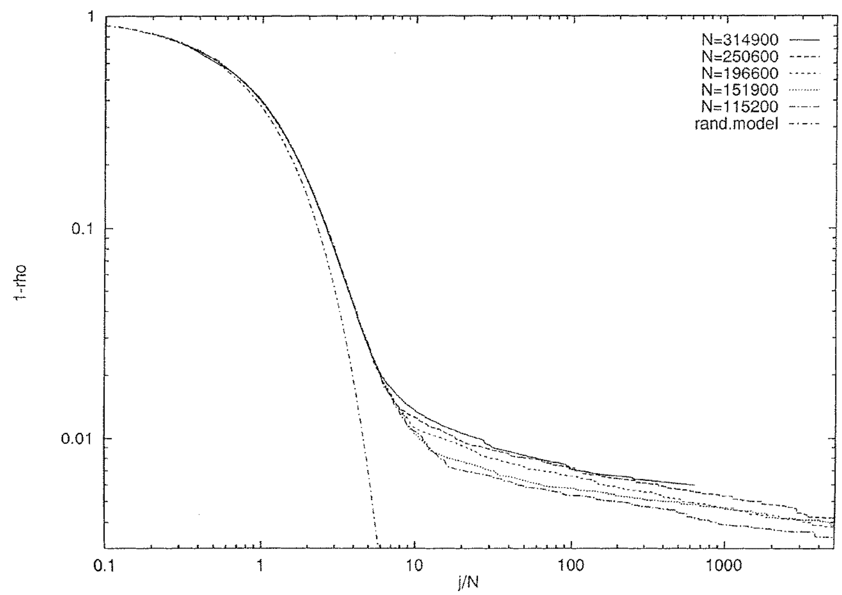

Figure 3. $1-\rho(j)$ against j/N on a log–log plot, for the 3D billiard (Prosen 1997a, b, $a = -0.1$, $b = 0$), for five different values of discretization N, and also for comparison the theoretical curve of the random model, namely $\exp(-j/N)$. It can be seen that $\gamma \approx 0.13$.

In fact, to be precise, we have drawn in figures 2 and 4 not the raw data for $w(g)$ but in fact their values after deconvolution: we have identified the maximum peak of the raw data with the value $g = 1$, subtracted from the peak the smooth background and deconvolved the peak data with the Poissonian spreading function. The final result, therefore, is a smooth background of $w(g)$ for $g < 1$ and a delta spike at $g = 1$.

In conclusion, we have developed the *general Poissonian model* or *the generalized random model*, which allows for different *a priori* probabilities for visiting cells in the discretized phase space or surface of section. The *a priori* probabilities are proportional to the so-called greyness parameter g of the cells, and their distribution is the greyness distribution $w(g)$, which tells us how many cells there are in the interval $(g, g + \mathrm{d}g)$. The consequence of our model is the universal scaling law, which states that the relative discrete measure $\rho(j)$ of the occupied cells on the chaotic component is a function of j/N only, and thus does not depend separately on j and N. We have also shown that the relative Lebesgue measure μ of the chaotic component is given by the integral $\mu = \int_0^1 g w(g)\,\mathrm{d}g$. In 2D systems $w(g)$ goes to $\delta(1 - g)$, such that typically the smooth background goes to zero as a power law $w_N(g) \propto N^{-f}$, where the exponent f is $2-d$ and d is the fractal dimension of the boundary

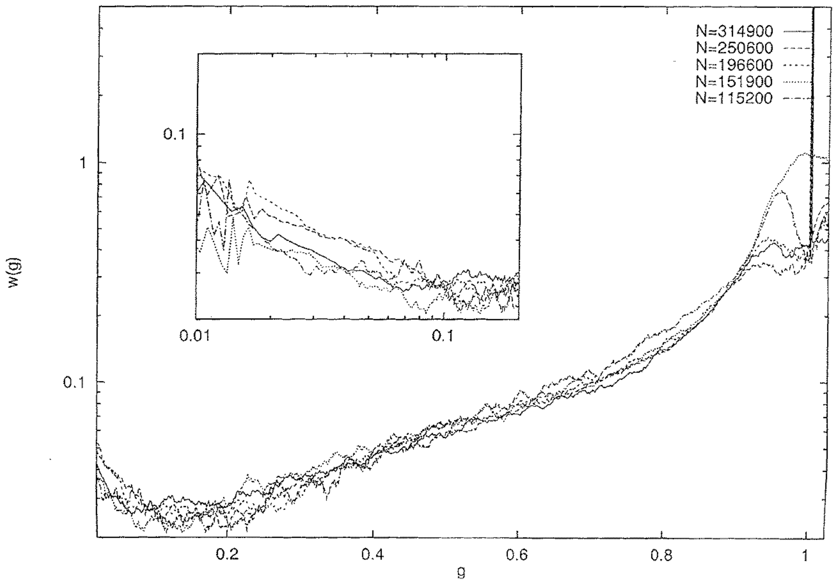

Figure 4. $w(g)$ for the same system as in figure 3, at the same five different values of N, after deconvolution. The smooth background of $w(g)$ clearly converges to a non-zero limiting value with increasing N. See the text for details. In the inset we show the same curves on a log–log plot. It can be deduced that $\beta \approx 0.48$.

of the chaotic component. Grey cells $g < 1$ in 2D systems are namely those cells that lie on the boundary of the chaotic region. In systems with three or more degrees of freedom we have a non-trivial $w(g)$, even in the limit $N \to \infty$, and thus $w(g)$ is a certain 'signature' of the chaotic region. We have also given estimates of the expected statistical error. Thus, we believe that our present work gives an excellent description of the diffusion in chaotic regions of Hamiltonian systems with mixed dynamics, especially in KAM-type systems.

Another type of generalization of the random model will be dealt with in our next work, namely the case of weakly-coupled chaotic components, where for each component we can assume the applicability of the general Poissonian model. Examples include, for example, weakly-coupled ergodic 2D billiards.

Both generalizations of the random model (Robnik *et al* 1997, paper I) are improvements in the understanding of the transport processes in Hamiltonian systems (MacKay *et al* 1984).

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